Zero-One Laws for Hypercyclicity

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March, 2014

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Let $p \geq 1$, and $B : \ell^p \to \ell^p$ be the unilateral backward shift defined by $B(a_0, a_1, a_2, \ldots) = (a_1, a_2, a_3, \ldots)$.

- Rolewicz (1969): If $t \in (1, \infty)$, then there exists a vector x in ℓ^p so that $\{x, (tB)x, (tB)^2x, (tB)^3x, \ldots\}$ is dense in ℓ^p.

Hypercyclicity Criterion

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- Kitai (1982), Gethner and Shapiro (1987): $T : X \to X$ is hypercyclic if there is a dense set D of X and T has a right inverse S so that $T^n x \to 0$ and $S^n x \to 0$ for each vector $x \in D$.

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- Read (1989): There is an operator T on ℓ^1 with no nontrivial closed invariant subset. That is, every nonzero vector x has the property that $\overline{\text{orb}(T,x)} = \ell^1$.

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If X is a Hilbert space, no normal operator is hypercyclic.

Hypercyclic vectors

Suppose $\{x_j : j \geq 1\}$ is a countable dense subset of X, and x is a vector in X. For x to be a hypercyclic vector, the following must hold:

For all x_j and for all $\epsilon > 0$, there is a integer n such that $\|T^n x - x_j\| < \epsilon$;

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Taking $\epsilon = 1/k$ in the above, we have

$$\mathcal{HC}(T) = \bigcap_{j,k=1}^{\infty} \bigcup_{n=1}^{\infty} T^{-n}B\left(x_j, \frac{1}{k}\right).$$

A Basic Zero-One Law for Hypercyclic Vectors

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Baire Category Theorem $\Longrightarrow$

If $\{T_n : X \to X | n \geq 1\}$ is a countable collection of hypercyclic operators, then their set of *common hypercyclic vectors*

$$\bigcap_{n=1}^{\infty} \mathcal{HC}(T_n)$$

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- Salas (1999): If B is the unilateral backward shift, is the set of common hypercyclic vectors

$$\bigcap_{t>1} \mathcal{HC}(tB) \neq \phi?$$

- Abakumov & Gordon (2003): $\bigcap_{1<t} \mathcal{HC}(tB) =$ the set of common hypercyclic vectors for tB is a dense G_δ set.

Existence of a G_δ Set of Common Hypercyclic Vectors

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- with Sanders (2009): Reproved the same result with a simpler proof by introducing an easier sufficient condition for common hypercyclicity that generalizes the Hypercyclicity Criterion for a path of operators.

  If I is an interval, and $F : I \to B(X)$ is said to be a <u>*path of operators*</u> if F is a continuous map with respect to the norm topology of $B(X)$ and the usual topology of I.

$T : \ell^p \to \ell^p$ is said to be a *unilateral weighted backward shift* if there is a bounded positive weight sequence $\{w_j : j \geq 1\}$ such that

$$T(a_0, a_1, a_2, \ldots) = (w_1 a_1, w_2 a_2, w_3 a_3, \ldots).$$

Unilateral Weighted Backward Shifts on ℓ^p

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- Grosse-Erdmann (2000): Generalizations to Fréchet spaces.

Bilateral Weighted Shifts on ℓ^p

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- Salas (1995): A bilateral weighted shift T is hypercyclic if and only if for any $\epsilon > 0$, and $q \in \mathbb{N}$, there is an arbitrarily large n such that whenever $|k| \leq q$,

$$\prod_{j=1}^{n} w_{k+j} > \frac{1}{\epsilon} \quad \text{and} \quad \prod_{j=0}^{n-1} w_{k-j} < \epsilon.$$

- with Sanders (2009): Between any two hypercyclic unilateral weighted backward shifts, there is a path of such operators with a dense G_δ set of common hypercyclic vectors. Also, there is a path of such operators with no common hypercyclic vectors.

Paths of Hypercyclic Weighted Shifts on ℓ^p

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Natural Question: Can we have "a lot" of operators in a path and yet their common hypercyclic vectors still form a dense G_δ subset? What do we mean by "a lot"?

Existence of Hypercyclic Operators

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Definition. An operator on X is said to be *chaotic* if and only if it is hypercyclic and has a dense set of periodic points.

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- Bonet & Martínez-Giménez & Peris (2001): There is a separable, infinite dimensional Banach space which admits no chaotic operator.

SOT = Strong Operator Topology of the operator algebra $B(X)$.

- (2002): For a separable, infinite dimensional Hilbert space H, the hypercyclic operators on H are SOT-dense in $B(H)$.

A Zero-One Law for Chaotic Operators

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- (2002): For a separable, infinite dimensional Hilbert space H, the hypercyclic operators on H are SOT-dense in $B(H)$.
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  Indeed, if $T \in B(X)$ is hypercyclic, then its conjugate class, or similarity orbit, $\{A^{-1}TA : A \text{ invertible on } X\}$ is SOT-dense in $B(X)$.

A Double Density Theorem

Let H be separable, infinite dimensional Hilbert space over $\mathbb{C}$.

- with Sanders (2011): There is a path of chaotic operators in $B(H)$ that is SOT-dense in $B(H)$, and each operator on the path shares the exact same set $\mathcal{G}$ of common hypercyclic vectors.

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- Corollary: The path can be taken so that each operator along the path satisfies the hypercyclicity criterion.
- Corollary: The hypercyclic operators in $B(H)$ are SOT-connected.
- Corollary: The hypercyclic operators T in $B(H)$ with $\mathcal{G} \subset \mathcal{HC}(T)$ are SOT-connected.

Similarity Orbits

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Observations of some zero-one phenomenon:

(1) If $\mathcal{HC}(T) = X \setminus \{0\}$, the set of common hypercyclic vectors for $\mathcal{S}(T)$ is also $X \setminus \{0\}$.

(2) If $\mathcal{HC}(T) \neq X \setminus \{0\}$, the set of common hypercyclic vectors for $\mathcal{S}(T)$ is empty.

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- with Sanders (2012): If $T \in B(H)$ be hypercyclic, then $\mathcal{U}(T)$ contains a path $\mathcal{P}$ of operators so that $\overline{\mathcal{P}}^{SOT}$ contains $\mathcal{U}(T)$ and the common hypercyclic vectors for $\mathcal{P}$ is a dense G_δ set.

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Corollary: T is not hypercyclic iff every orb$(T, x) \cup \{0\}$ is closed.

A Zero-One Law for Orbital Limit Points

- Bourdon & Feldman (2003): If an orbit orb(T, x) is somewhere dense in a Banach space X then the orbit orb(T, x) is everywhere dense.

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Remark: (A), (B), (D) are equivalent for bilateral weighted shifts.

- with Sanders (2004): A unilateral weighted backward shift is hypercyclic if and only if it is weakly hypercyclic. But, there is a bilateral weighted shift that is weakly hypercyclic but not hypercyclic.

A Remark on Theorem

If $\text{orb}(T, x)$ has a nonzero limit point, we can only conclude T is hypercyclic but we cannot conclude that x is a hypercyclic vector, and in fact not even a cyclic vector.

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  (1) the weight $(w_j)_{j=1}^{\infty}$ of T is bounded below, and

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There are examples to show both (1) and (2) are needed for x to be a cyclic vector.

Proof of "(B) $\Longrightarrow$ (A)"

Suppose there exist a vector $x = (x_0, x_1, x_2, \ldots) \in \ell^p$ and a non-zero vector $f = (f_0, f_1, f_2, \ldots) \in \ell^p$ such that f is a limit point of the orbit $\text{Orb}(T, x)$.

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Since $f_j \neq 0$ for some $j \geq 0$, we assume without loss of generality that $f_0 \neq 0$. Hence there exist an increasing sequence $\{n_k : k \geq 1\} \subset \mathbb{N}$ and an integer $N > 0$ such that

$$\|T^{n_k}x - f\| < \frac{1}{2^k} < \frac{|f_0|}{2},$$

for all $k \geq N$. Then

$$T^{n_k}x = T^{n_k}(x_0, x_1, x_2, \ldots) = (w_1 \cdots w_{n_k} x_{n_k}, \ldots).$$

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Hence $\|T^{n_k}x - f\| \geq |w_1 \cdots w_{n_k} x_{n_k} - f_0|$. So there exists a sequence $\{n_k : k \geq 1\}$ such that

$$|w_1 \cdots w_{n_k} x_{n_k} - f_0| < |f_0|/2,$$

for all $k \geq N$.

Thus $|f_0|/2 < |w_1 \cdots w_{n_k} x_{n_k}|$ and so $\frac{|f_0|}{2(w_1 \cdots w_{n_k})} < |x_{n_k}|$ for all $k \geq N$. Hence we get that

$$\frac{|f_0|^p}{2^p (w_1 \cdots w_{n_k})^p} < |x_{n_k}|^p, \text{ for all } k \geq N \; .$$

Now since $x \in \ell^p$ we have

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It follows that $\frac{1}{(w_1 \ldots w_{n_k})^p} \to 0$. That is, there exists an increasing sequence $\{n_k\}$ such that $w_1 \cdots w_{n_k} \to \infty$ as $k \to \infty$.

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Thus by Salas' criterion for hypercyclicity of unilateral backward shifts that $\sup\{w_1 \cdots w_n : n \geq 1\} = \infty$, we have that T is hypercyclic. □

Recall: A Zero-One Law for Orbital Limit Points

- with Seceleanu (2012): Let $T : \ell^p \to \ell^p$ be a unilateral weighted backward shift. The following are equivalent:

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Proof of "(C) $\implies$ (B)"

Let $x = (x_0, x_1, x_2, \ldots) \in \ell^p$ be a vector whose $\mathrm{Orb}(T, x)$ has $f = (f_0, f_1, f_2, \ldots) \in \ell^p$ as a non-zero weak limit point, with $f_k \neq 0$.

Considering the weakly open sets that contain f, we get that for all $j \geq 1$ there exists an $n_j \geq 1$ such that $|\langle T^{n_j} x - f, e_k \rangle| < \frac{1}{j}$.

That is $\left| w_{k+1} \cdots w_{k+n_j} x_{k+n_j} - f_k \right| < \frac{1}{j}$, for all $j \geq 1$.

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Next, we inductively pick a subsequence $\{n_{j_k}\}$ of $\{n_j\}$ as follows:
1. Let $j_1 = 1$.
2. Once we have chosen j_m we pick $j_{m+1} > j_m$ such that

$$k + n_{j_m} < n_{j_{m+1}} \text{ and } \sum_{i=j_{m+1}}^{\infty} |x_{k+n_i}|^p \leq \frac{1}{j_m \cdot \|T\|^{p \cdot n_{j_m}}}.$$

Thus we can assume, by taking a subsequence if necessary, that

$$\{n_j\} \text{ satisfies } k + n_j < n_{j+1} \text{ and } \sum_{i=j+1}^{\infty} |x_{k+n_i}|^p \leq \frac{1}{j \cdot \|T\|^{p \cdot n_j}}.$$

Let $y = \sum_{i=1}^{\infty} x_{k+n_i} \cdot e_{k+n_i}$. Clearly y is in ℓ^p, because x is.

Then $T^{n_m} y = \sum_{i=1}^{\infty} x_{k+n_i} \cdot T^{n_m} e_{k+n_i}$. But $k + n_i < n_{i+1}$ for all $i \geq 1$ and so $k + n_i < n_m$ for all $i < m$. Thus since T is a unilateral backward shift we conclude that $T^{n_m} y = \sum_{i=m}^{\infty} x_{k+n_i} \cdot T^{n_m} e_{k+n_i}$.

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Furthermore, since the vectors $T^{n_m} e_{k+n_i}$ and $T^{n_m} e_{k+n_j}$ have disjoint support for $i \neq j$, that is $\widehat{T^{n_m} e_{k+n_i}}(s) = 0$ whenever $\widehat{T^{n_m} e_{k+n_j}}(s) \neq 0$, we have that

$$\begin{aligned} & \| T^{n_m} y - f_k e_k \| \\ \leq \quad & \| (w_{k+1} \cdots w_{k+n_m} x_{k+n_m} - f_k) \cdot e_k \| + \left\| \sum_{i=m+1}^{\infty} x_{k+n_i} \cdot T^{n_m} e_{k+n_i} \right\| \end{aligned}$$

Thus,

$$
\begin{aligned}
& \|T^{n_m}y - f_k e_k\| \\
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\leq \; & \frac{1}{m} + \left[\sum_{i=m+1}^{\infty} |x_{k+n_i}|^p \cdot \|T\|^{p\cdot n_m}\right]^{1/p} \leq \frac{1}{m} + \frac{1}{\sqrt[p]{m}} \to 0 \quad as \;\; m \to \infty.
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Thus $T^{n_m}y \to f_k e_k$ in norm as $m \to \infty$, where $f_k e_k \neq 0$ in ℓ^p, and hence $\mathrm{Orb}(T, y)$ has a non-zero limit point. □

Bergman Spaces

Let Ω be a region in $\mathbb{C}$ and $H^\infty(\Omega)$ be the algebra of all bounded analytic functions on Ω.

Let $A^2(\Omega) = \{f : \Omega \to \mathbb{C} \,|\, f \text{ analytic, and } \int_\Omega |f|^2 \, dA < \infty\}$ be the Bergman space.

If $\varphi \in H^\infty(\Omega)$, then we define $M_\varphi : A^2(\Omega) \to A^2(\Omega)$ by $M_\varphi f = \varphi f$.

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- Godefroy & Shapiro (1991): The adjoint operator $M_\varphi^* : A^2(\Omega) \to A^2(\Omega)$ is hypercyclic if and only if $\varphi(\Omega)$ intersects the unit circle.

A Zero-One Law for Adjoint Multiplication Operators

Let $\varphi \in H^\infty(\Omega)$ be a nonconstant function, and
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What about the Hardy Space?

Let $\mathbb{D}$ be the open unit disk, and let

$$H^2 = \left\{ f : \mathbb{D} \to \mathbb{D} \,|\, f(z) = \sum_0^\infty a_n z^n \text{ analytic and } \sum_0^\infty |a_n|^2 < \infty \right\}$$

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- with Seceleanu (2012): If $\alpha > 0$ is an irrational number, and $\varphi(z) = e^{2\pi i \alpha} z$, then C_φ has an orbit with the identity function $\psi(z) \equiv z$ as a nonzero limit point, but C_φ is not hypercyclic.